UNIT-IV

Rank of a Matrix:

Example 1.1

Find the rank of the matrix $\begin{pmatrix} 1 & 5 \\ 3 & 9 \end{pmatrix}$

Solution:

Let A= $\begin{pmatrix} 1 & 5 \\ 3 & 9 \end{pmatrix}$

Order of A is 2×2 $\therefore$ $\rho(A) \leq 2$

Consider the second order minor

$$\begin{vmatrix} 1 & 5 \\ 3 & 9 \end{vmatrix} = -6 \neq 0$$

There is a minor of order 2, which is not zero. $\therefore \rho(A) = 2$

Example 1.2

Find the rank of the matrix $\begin{pmatrix} -5 & -7 \\ 5 & 7 \end{pmatrix}$

Solution:

Let A= $\begin{pmatrix} -5 & -7 \\ 5 & 7 \end{pmatrix}$

Order of A is 2×2 $\therefore \rho(A) \leq 2$

Consider the second order minor

$$\begin{vmatrix} -5 & -7 \\ 5 & 7 \end{vmatrix} = 0$$

Since the second order minor vanishes, $\rho(A) \neq 2$

Consider a first order minor $|-5| \neq 0$

There is a minor of order 1, which is not zero

$\therefore \rho(A) = 1$

Example 1.3

Find the rank of the matrix $\begin{pmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{pmatrix}$

Solution:

Let A= $\begin{pmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{pmatrix}$

Order Of A is 3x3

$\therefore \rho(A) \leq 3$

Consider the third order minor $\begin{vmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{vmatrix} = 6 \neq 0$

There is a minor of order 3, which is not zero

$\therefore \rho(A) = 3.$

Example 1.4

Find the rank of the matrix $\begin{pmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{pmatrix}$

Solution:

Let A= $\begin{pmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{pmatrix}$

Order Of A is 3x3

$\therefore \rho\ (A) \leq 3$

Consider the third order minor $\begin{vmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{vmatrix} = 0$

Since the third order minor vanishes, therefore $\rho(A) \neq 3$

Consider a second order minor $\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 7 \neq 0$

There is a minor of order 2, which is not zero.

$\therefore \rho(A) = 2$.

Example 1.5

Find the rank of the matrix $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

Order of A is 3×4

$\therefore \rho(A) \leq 3$.

Consider the third order minors

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & 6 & 3 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \\ 3 & 3 & -7 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & -2 \\ 3 & 6 & -7 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ 6 & 3 & -7 \end{vmatrix} = 0$$

Since all third order minors vanishes, $\rho(A) \neq 3$.

Now, let us consider the second order minors,

Consider one of the second order minors $\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 6 \neq 0$

There is a minor of order 2 which is not zero.

$\therefore \rho(A) = 2$.

Echelon form and finding the rank of the matrix (upto the order of 3×4) : Solved Example Problems

Example 1.6

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$$

Find the rank of the matrix A=

Solution :

The order of A is 3×3.

$\therefore \rho(A) \leq 3$.

Let us transform the matrix A to an echelon form by using elementary transformations.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$	$R_2 \rightarrow R_2 - 2R_1$ $R_3 \rightarrow R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$	$R_3 \rightarrow R_3 - R_2$
The above matrix is in echelon form	

The number of non zero rows is 2

$\therefore$Rank of A *is* 2.

$\rho(A) = 2$.

Note

A row having atleast one non -zero element is called as non-zero row.

Example 1.7

Find the rank of the matrix A= $\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$

Solution:

The order of *A* is 3 × 4.

∴ $\rho(A) \le 3$.

Let us transform the matrix *A* to an echelon form

Matrix *A*	Elementary Transformation
$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	
$A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	$R_1 \leftrightarrow R_2$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{pmatrix}$	$R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$	$R_3 \to R_3 + 5R_2$

The number of non zero rows is 3. ∴ $\rho(A) = 3$.

Example 1.8

Find the rank of the matrix A= $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{pmatrix}$

Solution:

The order of *A* is 3 × 4.

∴ $\rho(A) \leq 3$.

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -2 \end{pmatrix}$	$R_2 \rightarrow R_2 - 3R_1$ $R_3 \rightarrow R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$R_3 \rightarrow R_3 - R_2$

The number of non zero rows is 3.

∴ $\rho(A) = 3$.

Testing the consistency of non homogeneous linear equations (two and three variables) by rank method : Solved Example Problems

Example 1.9

Show that the equations $x + y = 5$, $2x + y = 8$ are consistent and solve them.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

AX=B

Matrix A	Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ $\sim \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 5 \\ 2 & 1 & 8 \end{pmatrix}$ $\sim \begin{pmatrix} 1 & 1 & 5 \\ 0 & -1 & -2 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\rho(A) = 2$	$\rho([A,B]) = 2$	

Number of non-zero rows is 2.

ρ (A)= ρ ([A, B]) = 2 = Number of unknowns.

The given system is consistent and has unique solution.

Now, the given system is transformed into

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$x + y = 5$

$y = 2$

$\therefore (1) \Rightarrow x + 2 = 5$

$x = 3$

Solution is $x = 3$, $y = 2$

Example 1.10

Show that the equations $2x + y = 5$, $4x + 2y = 10$ are consistent and solve them.

Solution:

The matrix equation corresponding to the system is

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

$$A \quad X = B$$

Matrix A	Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 5 \\ 4 & 2 & 10 \end{pmatrix}$	
$\sim\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$	$\sim\begin{pmatrix} 2 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$	$R_2 \rightarrow R_2 - 2R_1$
$\rho(A) = 1$	$\rho([A,B]) = 1$	

$\rho(A) = \rho([A, B]) = 1 <$ number of unknowns

∴ The given system is consistent and has infinitely many solutions.

Now, the given system is transformed into the matrix equation.

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$\Rightarrow 2x + y = 5$$

Let us take $y = k$, $k \in R$

$\Rightarrow 2x + k = 5$

$x = 1/2\,(5 - k)$

$x = 1/2\,(5 - k)$, $y = k$ for all $k \in R$

Thus by giving different values for k, we get different solution. Hence the system has infinite number of solutions.

Example 1.11

Show that the equations $3x - 2y = 6$, $6x - 4y = 10$ are inconsistent.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$$

$$AX = B$$

Matrix A	Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}$	$\begin{pmatrix} 3 & -2 & 6 \\ 6 & -4 & 10 \end{pmatrix}$	
$\sim\begin{pmatrix} 3 & -2 \\ 0 & 0 \end{pmatrix}$	$\sim\begin{pmatrix} 3 & -2 & 6 \\ 0 & 0 & -2 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\rho(A) = 1$	$\rho([A,B]) = 2$	

$$\therefore \rho([A, B]) = 2, \qquad \rho(A) = 1$$

$$\rho(A) \neq \rho([A,B])$$

∴The given system is inconsistent and has no solution.

Example 1.12

Show that the equations $2x + y + z = 5$, $x + y + z = 4$, $x - y + 2z = 1$ are consistent and hence solve them.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix}$$

$$A \quad X = B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 2 & 1 & 1 & 5 \\ 1 & 1 & 1 & 4 \\ 1 & -1 & 2 & 1 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 4 \\ 2 & 1 & 1 & 5 \\ 1 & -1 & 2 & 1 \end{pmatrix}$	$R_1 \leftrightarrow R_2$
$\sim \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & 1 & -3 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 3 & 3 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$
$\rho(A) = 3, \ \rho([A, B]) = 3$	

Obviously the last equivalent matrix is in the echelon form. It has three non-zero rows.

ρ(*A*) = ρ(*[A, B]*)= 3 = Number of unknowns .

The given system is consistent and has unique solution.

To find the solution, let us rewrite the above echelon form into the matrix form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}$$

x + y + z = 4 (1)

y + z = 3 (2)

3z = 3 (3)

(3)⇒ $z = 1$

(2)⇒ $y = 3 - z = 2$

(1) ⇒ $x = 4 - y - z$

x=1

∴ $x = 1, y = 2, z = 1$

Example 1.13

Show that the equations $x + y + z = 6$, $x + 2y + 3z = 14$, $x + 4y + 7z = 30$ are consistent and solve them.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ 30 \end{pmatrix}$$

$$A \quad X = B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 4 & 16 \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_2$
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$
$\rho(A) = 2, \rho([A,B]) = 2$	

Obviously the last equivalent matrix is in the echelon form. It has two non-zero rows.

∴ $\rho([A, B]) = 2, \rho(A) = 2$

$\rho(A) = \rho([A, B]) = 2 <$ Number of unknowns.

The given system is consistent and has infinitely many solutions.

The given system is equivalent to the matrix equation,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix}$$

$x + y + z = 6$ (1)

$y + 2z = 8$ (2)

(2)$\Rightarrow y = 8 - 2z$,

(1)$\Rightarrow x = 6 - y - z = 6 - (8 - 2z) - z = z - 2$

Let us take $z = k$, $k \in R$, we get $x = k - 2$, $y = 8 - 2k$, Thus by giving different values for k we get different solutions. Hence the given system has infinitely many solutions.

Example 1.14

Show that the equations $x - 4y + 7z = 14$, $3x + 8y - 2z = 13$, $7x - 8y + 26z = 5$ are inconsistent.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 5 \end{pmatrix}$$

$$A \quad X = B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & -4 & 7 & 14 \\ 3 & 8 & -2 & 13 \\ 7 & -8 & 26 & 5 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 20 & -23 & -93 \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 7R_1$
$\sim \begin{pmatrix} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 0 & 0 & 64 \end{pmatrix}$	$R_3 \to R_3 - R_2$
$\rho(A) = 2, \ \rho([A,B]) = 3$	

The last equivalent matrix is in the echelon form. [*A*, *B*] has 3 non-zero rows and [*A*] has 2 non-zero rows.

$$\therefore \quad \rho([A, B]) = 3, \qquad \rho(A) = 2$$

$$\rho(A) \neq \rho([A, B])$$

The system is inconsistent and has no solution.

Example 1.15

Find k, if the equations $x + 2y - 3z = -2$, $3x - y - 2z = 1$, $2x + 3y - 5z = k$ are consistent.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & -2 \\ 2 & 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ k \end{pmatrix}$$

$$A \qquad X = B$$

Augmented matrix $[A,B]$	**Elementary Transformation**
$\begin{pmatrix} 1 & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & k \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & -3 & -2 \\ 0 & -7 & 7 & 7 \\ 0 & -1 & 1 & 4+k \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 2 & -3 & -2 \\ 0 & -7 & 7 & 7 \\ 0 & 0 & 0 & 21+7k \end{pmatrix}$	$R_3 \to 7R_3 - R_2$
$\rho(A) = 2, \rho([A, B]) = 2$ or 3	

For the equations to be consistent, $\rho([A, B]) = \rho(A) = 2$

$\therefore 21 + 7k = 0$

$7k = -21$.

$k = -3$

Example 1.16

Find k, if the equations $x + y + z = 7$, $x + 2y + 3z = 18$, $y + kz = 6$ are inconsistent.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & k \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 18 \\ 6 \end{pmatrix}$$

$$A \quad X = B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 7 \\ 1 & 2 & 3 & 18 \\ 0 & 1 & k & 6 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 11 \\ 0 & 1 & k & 6 \end{pmatrix}$	$R_2 \to R_2 - R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & k-2 & -5 \end{pmatrix}$	$R_3 \to R_3 - R_2$

For the equations to be inconsistent

ρ ($[A, B]$) $\neq \rho$ (A)

It is possible if $k - 2 = 0$.

K=2

Example 1.17

Investigate for what values of '*a*' and '*b*' the following system of equations

x + y + z = 6, x + 2 y + 3z = 10, x + 2 y + az = b have

(i) no solution (ii) a unique solution (iii) an infinite number of solutions.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \\ b \end{pmatrix}$$

$$A \quad X = B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & a & b \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & a-1 & b-6 \end{pmatrix}$	$R_2 \to R_2 - R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & a-3 & b-10 \end{pmatrix}$	$R_3 \to R_3 - R_1$

Case (i) For no solution:

The system possesses no solution only when $\rho(A) \neq \rho([A, B])$ which is possible only when $a - 3 = 0$ and $b - 10 \neq 0$

Hence for $a = 3$, $b \neq 10$, the system possesses no solution.

Case (ii) For a unique solution:

The system possesses a unique solution only when $\rho(A) = \rho([A, B])$ = number of unknowns.

i.e when $\rho(A) = \rho([A, B]) = 3$

Which is possible only when $a - 3 \neq 0$ and b may be any real number as we can observe .

Hence for $a \neq 3$ and $b \in R$, the system possesses a unique solution.

Case (iii) For an infinite number of solutions:

The system possesses an infinite number of solutions only when

$\rho(A) = \rho([A, B]) <$ number of unknowns

i,e when $\rho(A) = \rho([A, B]) = 2 < 3$ (number of unknowns) which is possible only when $a - 3 = 0$, $b - 10 = 0$

Hence for $a = 3$, $b = 10$, the system possesses infinite number of solutions.