Unit-I

Partial Differentiation

Let $z = f(x, y)$ be a function of two variables x and y.Then $\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$, if it exists, is said
to be partial derivative or partial differential coefficient of z or $f(x, y)$ w.r.t. x.It is denoted by the
symbol $\boldsymbol{\frac{\partial z}{\partial x}}$ ***or*** $\boldsymbol{\frac{\partial f}{\partial x}}$ ***or*** $\boldsymbol{f_x}$.
Similarly , the partial derivative of $z = f(x, y)$ w.r.t $'y'$ keeping $'x'$ as constant is defined as $\lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$, and is denoted by $\boldsymbol{\frac{\partial z}{\partial y}}$ ***or*** $\boldsymbol{\frac{\partial f}{\partial y}}$ ***or*** $\boldsymbol{f_y}$.

Note: if $z = f(x, y)$ then first order partial derivatives are $p = \frac{\partial z}{\partial x}$ $q = \frac{\partial z}{\partial y}$

Second order partial derivatives are $r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$

$$s = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

Chain rule of partial differentiation:

(i) If $z = f(x, y)$ where $x = \phi(t), y = \psi(t)$ then z is called a composite function of two variables

(ii)If $z = f(x, y)$ where $x = \phi(u, v), y = \psi(u, v)$ then z is called a composite function of two variables.

If $z = f(x, y)$ where $x = \phi(u, v), y = \psi(u, v)$ then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

these are referred as chain rule of partial differentiation.

Total derivative: If $z = f(x, y)$ where $x = \phi(t), y = \psi(t)$ then total derivative of z or total differential coefficient of z is denoted by $\frac{dz}{dt}$ and is defined as $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$.

Taylor's Theorem: If a function $f: [a, b] \to R$ is such that (i) $f^{(n-1)}$is continuous in $[a, b]$
(ii) $f^{(n-1)}$is derivable in (a, b)
and if $p \epsilon Z^+$, then there exists at least one c in (a, b) such that

$$f(b) = f(a) + \frac{b-a}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

Where $R_n = \frac{(b-a)^p (b-c)^{n-p}}{(n-1)! p} f^{(n)}(c)$ (Schlomilch-Roche's form of remainder)

$= \frac{(b-a)^n}{n!} f^{(n)}(c)$ (Lagrange's form of remainder)

$= \frac{(b-a)(b-a)^{(n-1)}}{(n-1)!} f^{(n)}(c)$ (Cauchy's form of remainder)

Maclaurin's theorem: : If a function $f: [0, x] \to R$ is such that (i) $f^{(n-1)}$is continuous in $[0, x]$
(ii) $f^{(n-1)}$is derivable in $(0, x)$
and if $p \epsilon Z^+$, then there exists at least one θ in $(0, x)$ such that

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n$$

Where $R_n = \frac{x^n(1-\theta)^{n-p}}{(n-1)!p}f^{(n)}(\theta x)$ (Schlomilch-Roche's form of remainder)

$= \frac{x^n}{n!}f^{(n)}(\theta x)$ (Lagrange's form of remainder)

$= \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\theta x)$ (Cauchy's form of remainder)

Taylor's series for functions of two variables: Taylor's series of a function $f(x,y)$ about the point (a,b) (or) in powers of (x-a) and (y-b) is given by

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \cdots$$

Maclaurin's series for functions of two variables: Maclaurin's series expansion of a function of two variables (or) series expansion n powers of x and y is given by

$$f(x,y) = f(0,0) + (x)f_x(0,0) + (y)f_y(0,0) + \frac{1}{2!}\left[(x)^2 f_{xx}(0,0) + 2(x)(y)f_{xy}(0,0) + (y)^2 f_{yy}(0,0)\right] + \cdots$$

JACOBIANS AND FUNCTONAL DEPENDANCE:

Jacobian: If u and v are functions of x, y then

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of *u*, *v* with respect to *x*, *y*.

If *u*, *v*, *w* are functions of *x*, *y*, *z* then

$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$ is called Jacobian of *u*, *v*, *w* with respect to *x*, *y*, *z*

Properties of Jacobians:

1) If *u*, *v* are functions of *r*, *s* and *r*, *s* are functions of *x*, *y*, then $\frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$.

2) If *u*, *v* are functions of *x*, *y*, then $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$.

Functional Dependence:

1) If u, *v* are functions of *x*, *y*, then *u*, *v* are functionally dependent if and only if $\frac{\partial(u,v)}{\partial(x,y)} = 0$.

2) If *u*, *v*, *w* are functions of *x*, *y*, *z* then *u*, *v*, *w* are functionally dependent if and only if $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES:

Definition: A function $f(x,y)$ is said to have a maximum or minimum at $x = a, y = b$, according as $f(a+h, b+k) < or > f(a,b)$,for all positive or negative small values of *h* and *k*.

The necessary and sufficient condition for a function $f(x, y)$ can not have a maximum or minimum at (a, b) are that $f_x(a, b) = 0, f_y(a, b) = 0$.

Working rule to find the maximum and minimum values of $f(x, y)$:

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$
2. Solve these simultaneous equations in x and y $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ .Let $(a, b), (c, d)$ ....be the pair of values obtained.
3. Calculate the values $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ for each pair of values.
4. (i) if $rt - s^2 > 0$ and $r < 0$ at $(a, b), f(a, b) is\ a\ maximum\ value$.
   (ii) if $rt - s^2 > 0$ and $r > 0$ at $(a, b), f(a, b) is\ a\ minimum\ value$.
   (iii) if $rt - s^2 < 0$ at $(a, b), f(a, b) is\ not\ an\ extreme\ value, i.e\ (a, b) is\ a\ saddle\ point$.
   (iv) if $rt - s^2 = 0$ at $(a, b), the\ case\ is\ dobtful\ and\ needs\ furthur\ investigation$.

Similarly examine the conditions for all other pair of values obtained.

LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

Sometimes it is required to find extreme values of a function of several variables which are not all independent but are connected by some relation. If it is possible we convert the given function in to function of two variable by using the given relations. If not We use Lagrange's method.

If we require extreme value of $f(x, y, z)$ $subject\ to\ the\ condition$ $\emptyset(x, y, z) = 0$

Working Rule:

1.Write $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$

2.Obtain the equations $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$.

3. Solve the above equations together with $\phi(x, y, z) = 0$.The values of x, y, z so obtained will give the stationary values of $f(x, y, z)$.