UNIT-II

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

Differential equation: An equation involving derivatives is called a differential equation.
Ordinary differential equation: A differential equation involving ordinary derivatives(derivatives of functions of single variable) is called an ordinary differential equation.

Ex: $\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}} + 2x\frac{dy}{dx} = y^2$

Partial differential equation: A differential equation involving partial derivatives(derivatives of functions of two or more variables) is called a partial differential equation.
Ex: $\frac{\partial^3 z}{\partial x^3} - 5x^2y\left(\frac{\partial z}{\partial y}\right)^2 = 2e^x\left(\frac{\partial z}{\partial x}\right)$

Order of a differential equation: The order of the highest order derivative involved in a differential equation is called the order of the differential equation.

Ex: The order of the differential equation $\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}} + 2x\frac{dy}{dx} = y^2$ is 2.

Degree of a differential equation: The degree of the differential equation is the degree of the highest ordered derivative appearing it,after the equation has been expressed in a form free from fractional powers .

Ex: The order of the differential equation $\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}} + 2x\frac{dy}{dx} = y^2$ is 3.

Formation of ordinary differential equation:
By eliminating arbitrary constants from a relation we can form the ordinary differential equation.
Working rule for the formation of ordinary differential equation:
1.Consider a relation of the form $f(x, y, c_1, c_2, \dots c_n) = 0$ where $c_1, c_2 \dots., c_n$ are n arbitrary constants.
2.Differentiate the given relation successively n times.
3.With the help of n relations obtained and the given relation eliminate the n arbitrary constants to get
the required differential equation.

Solution of a differential equation: A relation between dependent and independent variables which satisfies a differential equation is called a solution of the differential equation.
Ex: Consider $\frac{dy}{dx} = y$. for this equation ,$y = e^x, y = 3e^x, y = \frac{-2}{5}\ e^x$ are the solutions for all x.

General solution of a differential equation: A solution of a differential equation in which the number of arbitrary constants is equal to order of the differential equation is called general solution of a differential equation.
Ex: $y = ce^x$ is the general solution of the differential equation.

Particular solution of a differential equation: A solution of a differential equation which is obtained by giving particular values for the arbitrary constants in the general soution is called particular solution of the differential equation.

Ex: $y = e^x, y = 2e^x$ are the particular solutions of $\frac{dy}{dx} = y$.

Exact differential equation: A differential equation $Mdx + Ndy = 0$ be a first order and first degree differential equation ,Where M and N are real valued functions of x and y. Then the equation $Mdx + Ndy = 0$ is said to be exact differential eqution if there exists a function f(x,y) such that

$\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N \quad i.e \quad df = Mdx + Ndy.$

Note: Condition for Exactness is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Working rule to solve Exact Differential Equation:

Step I: Write the given differential equation in the form $Mdx + Ndy = 0$.

Step2:Verify the condition for exactness $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If it is exact go to next step.

Step3: The general solution is obtained from

$$\int M\, dx + \int (terms\ of\ N\ not\ containing\ x) dy = c$$

Integrating Factor: A non exact differential equation can be made exact by multiplying with a function of x and y say $f(x, y)$, then $f(x, y)$ is called an integrating factor of that differntial equation.

Ex: $\frac{1}{y^2}, \frac{1}{x^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}$ are all integrating factors of the diffential equation $ydx - xdy = 0$.

Equations reducible to Exact Differential Equations:

Type I: If $Mdx + Ndy = 0$ is a non exact differential equation, is a homogeneous differential equation and $Mx + Ny \neq 0$, then integrating factor = $\frac{1}{Mx+Ny}$

Type II: If $Mdx + Ndy = 0$ is a non exact differential equation, is of the form $yf(xy)dx + xg(xy)dy = 0$ and if $Mx - Ny \neq 0$, then integrating factor= $\frac{1}{Mx-Ny}$

Type III: If $Mdx + Ndy = 0$ is a non exact differential equation and if $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f(x)$, then

Integrating factor = $e^{\int f(x)dx}$.

Type IV: If $Mdx + Ndy = 0$ is anon exact differential equation and if $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = g(y)$, then Integrating factor =$e^{\int g(y)dy}$.

Type V: If $Mdx + Ndy = 0$ is a non exact differential equation and an integrating factor can be found by inspection.

Linear differential Equation: A differential equation in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together is called a linear differential equation.

Ex: $\frac{dy}{dx} + 7y = x^2$, $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - xy = 1$

Linear differential equation of the first order:

Any linear differential equation can be written in the form

$$\frac{dy}{dx}+Py=Q$$ Where P,Q are functions of x alone	Or	$$\frac{dx}{dy}+Px=Q$$ Where P,Q are functions of y alone
Working rule: **Step1**: Write the differential equation of the form $\frac{dy}{dx}+Py=Q$ **Step2**: Find I.F .,where I.F=$e^{\int P\,dx}$ **Step3**: The general solution is given by $$y(I.F)=\int Q(I.F)dx\,.$$		**Working rule:** **Step1:**Write the given differential equation in the form $\frac{dx}{dy}+Px=Q$ **Step2**:Find I.F., where $I.F=e^{\int P\,dy}$ **Step3**:The general solution is given by $$x(I.F)=\int Q(I.F)dy\,.$$

Differential Equations Reducible to Linear differential Equations of the First order:

Type-I Bernoulli's equation:

A differential equation of the form $$\frac{dy}{dx}+Py=Qy^n$$ Where P,Q are functions of x alone and nϵR Is called Bernoulli's differential equation in y		A differential equation of the form $$\frac{dx}{dy}+Px=Qx^n$$ Where P,Q are functions of y alone and nϵR Is called Bernoulli's differential equation in x
Working Rule: **Step1**:Write the differential equation is of the form $\frac{dy}{dx}+Py=Qy^n$ **Step2**:Divide the equation with y^n and substitute $y^{1-n}=t$ and hence $$y^{-n}\frac{dy}{dx}=\frac{1}{1-n}\frac{dt}{dx}$$ **Step3**:Solve the resulting differential equation ,which will be a linear differential equation in 't'. **Step4**: Replace 't' by y^{1-n} in the end to get required solution.		Working Rule: **Step1**:Write the differential equation is of the form $\frac{dx}{dy}+Px=Qx^n$ **Step2**:Divide the equation with x^n and substitute $x^{1-n}=t$ and hence $$x^{-n}\frac{dx}{dy}=\frac{1}{1-n}\frac{dt}{dy}$$ **Step3**:Solve the resulting differential equation ,which will be a linear differential equation in 't'. **Step4**: Replace 't' byx^{1-n} in the end to get required solution.

Type-2

Differential equations of the form $$f'(y)\frac{dy}{dx}+Pf(y)=Q$$ Where P,Q are functions of 'x' alone.		Differential equations of the form $$f'(x)\frac{dx}{dy}+Pf(x)=Q$$ Where P,Q are functions of 'y' alone.
Working Rule: Put f(y)=t and hence $f'(y)\frac{dy}{dx}=\frac{dt}{dx}$.		**Working Rule:** Put f(x)=t and hence $f'(x)\frac{dx}{dy}=\frac{dt}{dy}$.

Applications of First Order and First Degree Differential Equations:
Orthogonal Trajectories:

Def: A curve which cuts every member of a family of curves at right angle is called an orthogonal trajectory of that family of curves.

If every member of a family (I) of curves cuts at right angles of every member of another family (II) of curves, then the family (I) is said to be a family of orthogonal trajectories of family (II).Similarly family (II) is also a orthogonal trajectories of family (I).

Ex: Family of straight lines passing through origin is orthogonal trajectories of the family of circles at having centre's at origin.

Working Rule to Find Orthogonal Trajectories in Cartesian coordinates:

Step1:Form the differential equation corresponding to the given family of curves.

Step2:Replace $\frac{dy}{dx}$ $by\ -\frac{dx}{dy}$ in the obtained differential equation.

Step3:Solve the resulting differential equation to get the equation of the family of orthogonal trajectories of the given family of curves.

Working Rule to Find Orthogonal Trajectories in Polar coordinates:

Step1:Form the differential equation corresponding to the given family of curves.

Step2:Replace $\frac{dr}{d\theta}$ $by\ -r^2\frac{d\theta}{dr}$ in the obtained differential equation.

Step3:Solve the resulting differential equation to get the equation of the family of orthogonal trajectories of the given family of curves.

Newton's Law of Cooling: The rate of change in temperature of a body is directly proportional to difference of temparatures of the body and temperature of the surrounding medium.

$i.e\ \frac{d\theta}{dt} \propto (\theta-\theta_0) \quad \Rightarrow \frac{d\theta}{dt} = -k(\theta-\theta_0)$ ('-' indicates decrease in temperature)

On solving

$$\Rightarrow \theta = \theta_0 + ce^{-kt}$$

Here θ_0 is the temperature of the surrounding medium. θ is the temperature of the body at any time 't'.

Law of natural growth and decay: The rate of change in the amount of substance is directly proportional to the amount of substance present at any time t.

$$i.e\ \frac{dx}{dt} \propto x \quad \Rightarrow \frac{dx}{dt} = kx \ \Rightarrow x = ce^{kt} \quad (Law\ of\ natural\ growth)$$

$$\frac{dx}{dt} \propto x \quad \Rightarrow \frac{dx}{dt} = -kx \Rightarrow x = ce^{-kt}\ (Law\ of\ natural\ decay)$$