Unit-III

Linear Differential Equations of Second and Higher Order

Linear Differential Equations: A Differential Equation is said to be Linear if dependent variable and derivatives of dependent variable occur only in the first degree and are not multiplied together.

The general linear differential equation of the n th order is of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots . + P_n y = Q(x) \qquad \text{----------} (1)$$

Here $P_1, P_2, \ldots \ldots P_n, Q(x)$ are either functions of x or constants.

If $P_1, P_2, \ldots \ldots P_n$ are constants then the equation (1) is called Linear Differential equation with constant coefficients. The equation (1) is said to be homogeneous if $Q(x) = 0$ and non homogeneous if $Q(x) \neq 0$.

If $P_1, P_2, \ldots \ldots P_n$ are functions of x, then the equation (1) is called Linear differential equation with variable coefficients.

The complete solution of differential equation is given by $y = C.F + P.I$

Where C.F is called complementary function, P.I is called particular solution or particular integral.

The general solution of homogeneous equation $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots . + P_n y = 0$ is called complementary function.

Operator D: Denote $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \ldots$ with $D, D^2, D^3, \ldots$

The Linear Differential equation of n th order in operator form is

$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots . + P_n) y = Q(x)$ i.e. $f(D) y = Q(x)$

Rules for finding Complementary Function:

1. Consider the equation of the form $f(D)y = 0$
2. Write the auxiliary equation by replacing 'D' with 'm', 'y' with '1'.
   i.e. $f(m) = 0$
3. This represents polynomial equation in 'm' of degree 'n' and on solving it we get 'n' roots.

S.no	Roots of auxiliary equation $f(m) = 0$	Complementary function
1.	If the roots are real and distant i.e. $m_1, m_2, \ldots m_n$	C.F = $C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots C_n e^{m_n x}$
2.	If two roots are equal and rest are real and different i.e. $m_1, m_1, m_3 \ldots m_n$	C.F =$(C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_3 x} + \ldots + C_n e^{m_n x}$
3.	If three roots are real and equal and the rest are real and different i.e.$m_1, m_1, m_1, m_4 \ldots . m_n$	C.F =$(C_1 + C_2 x + C_3 x^2) e^{m_1 x} + C_4 e^{m_5 x} + \cdots + C_n e^{m_n x}$
4.	Two roots are complex say$\alpha + i\beta$, $\alpha - i\beta$ and the remaining are real and different.	C.F =$e^{\alpha x}(C_1 cos\beta x + C_2 sin\beta x) + C_3 e^{m_3 x} + \ldots + C_n e^{m_n x}$
5.	A pair of complex conjugate complex	C.F =$e^{\alpha x}[(C_1 + C_2 x) cos\beta x + (C_3 + C_4 x) sin\beta x]$

	roots $\alpha \pm i\beta$ repeated twice and the remaining roots are real and distinct	$+C_5 e^{m_5 x} + \cdots . + C_n e^{m_n x}$

Inverse operator: 1. $\frac{1}{D}\left(Q(x)\right) = \int Q(x)\, dx$

2. $\boldsymbol{\frac{1}{D-a}\left(Q(x)\right) = e^{ax} \int e^{-ax}\, Q(x)\, dx}$

3. $\frac{1}{D+a}\left(Q(x)\right) = e^{-ax} \int e^{ax}\, Q(x)\, dx$

Particular integral of $(D)y = Q(x)$:

Particular integral (P.I) = $\frac{1}{f(D)}\, Q(x)$

Rules For Finding Particular Integral in Special Cases:

1. If $Q = e^{ax}$ Where 'a' is constant.	P.I = $\frac{1}{f(D)}\, e^{ax} = \frac{1}{f(a)} e^{ax}\ , provided\ f(a) \neq 0$ $= \frac{xe^{ax}}{f^1(a)}\ , provided\ f(a) = 0, f^1(a) \neq 0$ $= x^2\, \frac{e^{ax}}{f^{11}(a)}\ , provided\ f(a) = 0, f^1(a) = 0, f^{11}(a) \neq 0$
2. If $Q = \sin ax\ or\ \cos ax$	P.I = $\frac{1}{f(D)} \sin ax = \frac{1}{\emptyset(D^2)} \sin ax = \frac{\sin ax}{\emptyset(-a^2)}\ \ if\ \emptyset(-a^2) \neq 0$ Similarly $= \frac{1}{f(D)} \cos ax = \frac{1}{\emptyset(D^2)} \cos ax = \frac{\cos ax}{\emptyset(-a^2)}\ \ if\ \emptyset(-a^2) \neq 0$ If $\emptyset(-a^2) = 0$ then use $\frac{1}{D^2+a^2} \sin ax = \frac{-x \cos ax}{2a}$ $\quad \frac{1}{D^2+a^2} \cos ax = \frac{x \sin ax}{2a}$
3.If $Q = x^k$, where k is constant	P.I=$\frac{1}{f(D)}\ x^k = \frac{1}{(1 \pm \emptyset(D))}\ x^k = (1 \pm \emptyset(D))^{-1}\, x^k$ Expand $(1 \pm \emptyset(D))^{-1}$in ascending powers using Binomial theorem up to the term containing D^k We use the following formulae $(1 + D)^{-1} = 1 - D + D^2 - D^3 + \cdots$ $(1 - D)^{-1} = 1 + D + D^2 + D^3 + \cdots$ $(1 + D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$ $(1 - D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$
4.If $Q = e^{ax}V$ Where V being a fn. of x	P.I=$\frac{1}{f(D)} (e^{ax}V) = e^{ax}\ \frac{1}{f(D+a)}\ (V)$
5.If $Q = x^m V$ V is either $\sin ax$ or $\cos ax$ only. If V is x^n P.I can be evaluated using (3). If V is e^{ax} P.I can be evaluated using (4).	P.I =$\frac{1}{f(D)} x^m \sin ax = Imaginary\ part\ of\ \frac{1}{f(D)}\ x^m e^{iax}$ $\frac{1}{f(D)} x^m \cos\ ax = real\ part\ of\ \frac{1}{f(D)}\ x^m e^{iax}$
6.If $Q = xV$ where V is a	P.I=$\frac{1}{f(D)}\ (xV) = \left[x - \frac{1}{f(D)} f^1(D)\right] \frac{1}{f(D)} V$

function of x	

Application:
LCR circuit:

Here L denotes Inductor of inductance L henrys. C denotes capacitor of capacitance C farads. R denotes Resistor of resistance R ohms. when these three are connected in series and connected to an electromotive force E(t) volts(a generator) as shown in the following figure.

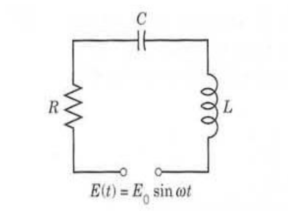

Fig. 60. *RLC*-circuit

Here the objective of this problem is when E, L, C, R values are given to find the current in circuit. Using Kirchhoff's voltage law" The voltage impressed on a closed loop is equal to the sum of the voltage drops across the elements of the loop.

When current 'I' flowing through a resistor or inductor or capacitor causes a voltage drop (voltage difference measured in volts)

Voltage drop for a resistor of resistance R is given by $I.R$

Voltage drop for a inductor of inductance L is given by $L\frac{dI}{dt}$

Voltage drop for a capacitor of capacitance C is given by $\frac{Q}{C}$ where Q coulombs is the charge on the capacitor related to current is given by $I = \frac{dQ}{dt} \Rightarrow Q = \int I(t)\, dt$

According to Kirchhoff's voltage law we get $L\frac{dI}{dt} + I.R + \frac{1}{C}\int I(t)\, dt = E(t)$

Where $E(t) = E_0 \sin \omega t$

Differentiate with respect to t

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E_0\omega\ cos\omega t$$

This shows that current in LCR circuit is obtained as the solution of the non homogeneous second order ODE with constant coefficients.

Example: Find steady state current in LCR circuit for the data

$$R = 8\ \Omega,\ L = 0.5\ \text{H},\ C = 0.1\ \text{F},\ E = 100 \sin 2t\ \text{V}$$

Solution: The differential equation for LCR circuit is $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E_0\omega\ cos\omega t$

$$0.5\frac{d^2I}{dt^2} + 8\frac{dI}{dt} + \frac{1}{0.1}I = 100.2\cos 2t$$